

Characterizing Frame Definability in Team Semantics via The Universal Modality

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Abstract. Let $\mathcal{ML}(\Box^+)$ denote the fragment of modal logic extended with the universal modality in which the universal modality occurs only positively. We characterize the definability of $\mathcal{ML}(\Box^+)$ in the spirit of the well-known Goldblatt-Thomason-Theorem. We show that an elementary class \mathbb{F} of Kripke frames is definable in $\mathcal{ML}(\Box^+)$ if and only if \mathbb{F} is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes. In addition, we initiate the study of modal frame definability in team-based logics. We show that, with respect to frame definability, the logics $\mathcal{ML}(\Box^+)$, modal logic with intuitionistic disjunction, and (extended) modal dependence logic all coincide. Thus we obtain Goldblatt-Thomason-style theorems for each of the logics listed above.

1 Introduction

Modal logic as a field has progressed far from its philosophical origin, e.g, from the study of the concepts of necessity and possibility. Modern modal logics are integral parts of both theoretical research and real life applications in various scientific fields such as mathematics, artificial intelligence, linguistics, economic game theory, and especially in many subfields of theoretical and applied computer science. Indeed, the general framework of modal logic has been found to be remarkably adaptive.

During the last decade there has been an emergence of vibrant research on modal and propositional logics with team semantics. The fundamental idea behind team semantics is crisp. The idea is to shift from points to sets of points as the satisfying elements of formulae. In ordinary Kripke semantics for modal logic, the formulae are evaluated on pointed models (\mathfrak{M}, w) , where \mathfrak{M} is a Kripke model and w is an element of the domain of \mathfrak{M} . In team semantics for modal logic the formulae are evaluated on pairs (\mathfrak{M}, T) , where \mathfrak{M} is an ordinary Kripke model and the set T , called a *team* of \mathfrak{M} , is a subset of the domain of \mathfrak{M} . This shift in semantics has no real effect if we only consider standard modal logic. The significance of this shift can be only seen once we extend modal logic with (a collection of) novel atomic formulae that state properties of teams.

In recent years multitude of different extensions of modal logic with novel atomic propositions on teams have been defined. The first of this kind was the modal dependence logic (\mathcal{MDL}) of Väänänen [20]. Modal dependence logic extends modal logic with *propositional dependence atoms*. A dependence atom, denoted by $\text{dep}(p_1, \dots, p_n, q)$, intuitively states that (inside a team) the truth value of the proposition q is functionally determined by the truth values of the propositions p_1, \dots, p_n . It was soon realized that \mathcal{MDL} lacks the ability to express temporal dependencies; there is no mechanism in \mathcal{MDL} to express dependencies that occur between different points of the model. This is due to the restriction that only proposition symbols are allowed in the dependence atoms of modal dependence logic. To overcome this defect Ebbing et al. [6] introduced the *extended modal dependence logic* (\mathcal{EMDL}) by extending the scope of dependence atoms to arbitrary modal formulae, i.e., dependence atoms in extended modal dependence logic are of the form $\text{dep}(\varphi_1, \dots, \varphi_n, \psi)$, where $\varphi_1, \dots, \varphi_n, \psi$ are formulae of modal logic. Subsequently multitude of related logics have been introduced.

The focus of the research on team-based logics has been in the computational complexity and expressive power. Hella et al. [11] established that exactly the properties of teams that are downward closed and closed under the so-called team k -bisimulation, for some finite k , are definable in \mathcal{EMDL} . In the article it was also shown that the expressive powers of \mathcal{EMDL} and $\mathcal{ML}(\otimes)$ (modal logic extended with intuitionistic disjunction) coincide. More recently Kontinen et al. have shown (in the manuscript [13]) that exactly the properties of teams that are closed under the team k -bisimulation are definable in the so-called *modal team logic*. These characterizations truly demonstrate the naturality of the related languages. For recent research related to computational complexity of modal dependence logics see, e.g., [6, 7, 12, 14, 15, 19]. The research related to proof theory has been less active, for related work see the PhD thesis [21] and the manuscript [17].

Modal logic extended with universal modality ($\mathcal{ML}(\Box)$) was first formulated by Goranko and Passy [10]. It extends modal logic by a novel modality \Box , called the universal modality, with the following semantics: the formula $\Box \varphi$ is true in a point w of a model \mathfrak{M} if φ is true in every point v of the model \mathfrak{M} . In this article we identify a connection between particular team-based modal logics and a fragment of $\mathcal{ML}(\Box)$. We will then use this connection in order to characterize frame definability of these team-based modal logics in the spirit of the well-known Goldblatt-Thomason Theorem.

The celebrated Goldblatt-Thomason Theorem [9] is a characterization of modal definability of elementary (i.e., first-order definable) classes of Kripke frames by four frame constructions: generated subframes, disjoint unions, bounded morphic images, and ultrafilter extensions. The theorem states that an elementary class of Kripke frames is definable by a set of modal formulae if and only if the class is closed under taking generated subframes, disjoint unions and bounded morphic images, and reflects ultrafilter extensions. The original proof of Goldblatt and Thomason was algebraic. A model-theoretic version of the proof was

later given by van Benthem [2]. From then on, Goldblatt-Thomason-style Theorems have been formulated for numerous extensions of modal logic such as modal logic with the universal modality [10], difference logic [8], hybrid logic [4], and graded modal logic [16].

This paper initiates the study of frame definability in the framework of team semantics. Our contribution is two-fold. Firstly, we give a Goldblatt-Thomason-style Theorem for a fragment of modal logic extended with universal modality. Secondly, we show that there is a surprising connection between this fragment and particular team-based modal logics. Let $\mathcal{ML}(\boxplus^+)$ denote the syntactic fragment of $\mathcal{ML}(\boxplus)$ in which the universal modality occurs only positively. We show that an elementary class of Kripke frames is definable in $\mathcal{ML}(\boxplus^+)$ if and only if it is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes. We then show that a class of frames is definable in $\mathcal{ML}(\boxplus^+)$ if and only if it is definable in $\mathcal{ML}(\boxtimes)$. We then continue by showing that, with respect to frame definability, \mathcal{MDL} and \mathcal{EMDL} coincide. From this observation and since, by the work of Hella et al. [11], the expressive powers of $\mathcal{ML}(\boxtimes)$ and \mathcal{EMDL} coincide, the above characterization of frame definability also holds for $\mathcal{ML}(\boxtimes)$, \mathcal{MDL} , and \mathcal{EMDL} .

The structure of the paper is as follows. In Section 2 we give a short introduction to modal logic extended with the universal modality and prove a normal form for $\mathcal{ML}(\boxplus^+)$. In Section 3 we first introduce the concept of frame definability. We then show that, with respect to frame definability, \mathcal{ML} , $\mathcal{ML}(\boxplus^+)$, and $\mathcal{ML}(\boxplus)$ form a strict hierarchy. In Section 4 we give a Goldblatt-Thomason-style characterization for the frame definability of $\mathcal{ML}(\boxplus^+)$. In Section 5 we introduce the team-based logics \mathcal{MDL} , \mathcal{EMDL} , and $\mathcal{ML}(\boxtimes)$. We then show that, with respect to frame definability, $\mathcal{ML}(\boxtimes)$, \mathcal{MDL} , \mathcal{EMDL} , and $\mathcal{ML}(\boxplus^+)$ coincide.

2 Modal Logic with Universal Modality

The syntax of modal logic with universal modality could be defined in any standard way. However in logics with team semantics, it is customary to assume that all formulae are in *negation normal form*, i.e., negations occur only in front of atomic propositions. In Section 5 we compare modal logic with universal modality to different logics with team semantics. In order to make these comparisons more straightforward, we define the syntax of standard modal logic also in negation normal form.

Let Φ be a set of atomic propositions. The set of formulae for *modal logic* $\mathcal{ML}(\Phi)$ is generated by the following grammar

$$\varphi ::= p \mid \neg p \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond \varphi \mid \Box \varphi, \quad \text{where } p \in \Phi.$$

The syntax of *modal logic with universal modality* $\mathcal{ML}(\boxplus)(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rules

$$\varphi ::= \boxplus \varphi \mid \boxtimes \varphi.$$

The syntax of *modal logic with positive universal modality* $\mathcal{ML}(\boxplus^+)(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rule $\varphi ::= \boxplus \varphi$. As usual, if the underlying set Φ of atomic propositions is clear from the context, we drop “ (Φ) ” and just write \mathcal{ML} , $\mathcal{ML}(\boxplus)$, etc. We also use the shorthands $\neg\varphi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$. By $\neg\varphi$ we denote the formula that can be obtained from φ by pushing all negations to the atomic level, and by $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$, we denote $\neg\varphi \vee \psi$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, respectively.

A (Kripke) *frame* is a pair $\mathfrak{F} = (W, R)$ where W , called the *domain* of \mathfrak{F} , is a non-empty set and $R \subseteq W \times W$ is a binary relation on W . By \mathbb{F}_{all} , we denote the class of all frames. We use $|\mathfrak{F}|$ to denote the domain of the frame \mathfrak{F} . A (Kripke) Φ -*model* is a tuple $\mathfrak{M} = (W, R, V)$, where (W, R) is a frame and $V : \Phi \rightarrow \mathcal{P}(W)$ is a valuation of the proposition symbols. The semantics of modal logic, i.e., the *satisfaction relation* $\mathfrak{M}, w \Vdash \varphi$, is defined via pointed Φ -models as usual. For the universal modality \boxplus and its dual \boxminus , we define

$$\begin{aligned} \mathfrak{M}, w \Vdash \boxplus \varphi &\Leftrightarrow \mathfrak{M}, v \Vdash \varphi, \text{ for every } v \in W, \\ \mathfrak{M}, w \Vdash \boxminus \varphi &\Leftrightarrow \mathfrak{M}, v \Vdash \varphi, \text{ for some } v \in W. \end{aligned}$$

If $\varphi \in \mathcal{ML}(\boxplus)(\Phi)$ is a Boolean combination of formulae beginning with \boxplus , we say that φ is *closed*. A formula set Γ is *valid in a model* $\mathfrak{M} = (W, R, V)$ (notation: $\mathfrak{M} \Vdash \Gamma$), if $\mathfrak{M}, w \Vdash \varphi$ holds for every $w \in W$ and every $\varphi \in \Gamma$. When Γ is a singleton $\{\varphi\}$, we simply write $\mathfrak{M} \Vdash \varphi$. We say that formulae φ_1 and φ_2 are *equivalent in Kripke semantics* ($\varphi_1 \equiv_K \varphi_2$), if the equivalence $\mathfrak{M}, w \Vdash \varphi_1 \Leftrightarrow \mathfrak{M}, w \Vdash \varphi_2$ holds for every model $\mathfrak{M} = (W, R, V)$ and every $w \in W$.

We will next define a normal form for $\mathcal{ML}(\boxplus^+)$. This normal form is a modification of the normal form for $\mathcal{ML}(\boxplus)$ by Goranko and Passy in [10].

- Definition 1.** (i) A formula φ is a *disjunctive \boxplus -clause* (conjunctive \boxplus -clause) if there exists a natural number $n \in \omega$ and formulae $\psi, \psi_1, \dots, \psi_n \in \mathcal{ML}$ such that $\varphi = \psi \vee \boxplus \psi_1 \vee \dots \vee \boxplus \psi_n$ ($\varphi = \psi \wedge \boxplus \psi_1$).
- (ii) A formula φ is in *conjunctive \boxplus -form* (disjunctive \boxplus -form) if φ is a conjunction (disjunction) of disjunctive \boxplus -clauses (conjunctive \boxplus -clauses).
- (iii) A formula φ is in \boxplus -form if φ is either in conjunctive \boxplus -form or in disjunctive \boxplus -form.

It is easy to show that for each $\mathcal{ML}(\boxplus^+)$ -formula in conjunctive \boxplus -form there exists an equivalent $\mathcal{ML}(\boxplus^+)$ -formula in disjunctive \boxplus -form, and vice versa. The proof of the following theorem can be found in Appendix A (Theorem A.1).

Theorem 1. For each $\mathcal{ML}(\boxplus^+)$ -formula φ , there exists a $\mathcal{ML}(\boxplus^+)$ -formula ψ in \boxplus -form such that $\varphi \equiv_K \psi$.

3 Modal Frame Definability

In this section, we first introduce the basic notions and results concerning frame definability used later on in the paper. We will then compare \mathcal{ML} , $\mathcal{ML}(\boxplus^+)$, and $\mathcal{ML}(\boxplus)$ with respect to frame definability.

Below we assume only that the logics \mathcal{L} and \mathcal{L}' are such that the global satisfaction relation for Kripke models (i.e., $\mathfrak{M} \Vdash \varphi$) is defined. A set Γ of \mathcal{L} -formulae is *valid in a frame* \mathfrak{F} (written: $\mathfrak{F} \Vdash \Gamma$) if $(\mathfrak{F}, V) \Vdash \varphi$ for every valuation $V : \Phi \rightarrow \mathcal{P}(W)$ and every $\varphi \in \Gamma$. A set Γ of \mathcal{L} -formulae is *valid in a class* \mathbb{F} of frames (written: $\mathbb{F} \Vdash \Gamma$) if $\mathfrak{F} \Vdash \Gamma$ for every $\mathfrak{F} \in \mathbb{F}$. Given a set Γ of \mathcal{L} -formulae, $\mathbb{FR}(\Gamma) := \{\mathfrak{F} \in \mathbb{F}_{\text{all}} \mid \mathfrak{F} \Vdash \Gamma\}$. We say that Γ *defines* the class $\mathbb{FR}(\Gamma)$. When Γ is a singleton $\{\varphi\}$, we simply say that φ defines the class $\mathbb{FR}(\Gamma)$. A class \mathbb{F} of frames is \mathcal{L} -*definable* if there exists a set Γ of \mathcal{L} -formulae such that $\mathbb{FR}(\Gamma) = \mathbb{F}$.

Definition 2. A class $\mathbb{C} \subseteq \mathbb{F}_{\text{all}}$ is *elementary* if there exists a set of first-order sentences with equality of the vocabulary $\{R\}$ that defines \mathbb{C} .

Definition 3. We write $\mathcal{L} \leq_F \mathcal{L}'$ if every \mathcal{L} -definable class of frames is also \mathcal{L}' -definable. We write $\mathcal{L} =_F \mathcal{L}'$ if both $\mathcal{L} \leq_F \mathcal{L}'$ and $\mathcal{L}' \leq_F \mathcal{L}$ hold and write $\mathcal{L} <_F \mathcal{L}'$ if $\mathcal{L} \leq_F \mathcal{L}'$ but $\mathcal{L}' \not\leq_F \mathcal{L}$.

It is easy to see that $\mathcal{ML} \leq_F \mathcal{ML}(\sqcup^+) \leq_F \mathcal{ML}(\sqcup)$. To show that the two occurrences of \leq_F here are strict, let us introduce two frame constructions.

Definition 4 (Disjoint Unions). Let $\{\mathfrak{F}_i \mid i \in I\}$ be a pairwise disjoint family of frames, where $\mathfrak{F}_i = (W_i, R_i)$. The disjoint union $\biguplus_{i \in I} \mathfrak{F}_i = (W, R)$ of $\{\mathfrak{F}_i \mid i \in I\}$ is defined by $W = \bigcup_{i \in I} W_i$ and $R = \bigcup_{i \in I} R_i$.

Definition 5 (Generated Subframes). Given any two frames $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$, \mathfrak{F}' is a *generated subframe* of \mathfrak{F} if (i) $W' \subseteq W$, (ii) $R' = R \cap (W')^2$, (iii) $w' R v'$ implies $v' \in W'$, for every $w' \in W'$. We say that \mathfrak{F}' is the *generated subframe of \mathfrak{F} by $X \subseteq |\mathfrak{F}|$* (notation: \mathfrak{F}_X) if \mathfrak{F}' is the smallest generated subframe of \mathfrak{F} whose domain contains X . \mathfrak{F}' is a *finitely generated subframe of \mathfrak{F}* if there is a finite set $X \subseteq |\mathfrak{F}|$ such that \mathfrak{F}' is \mathfrak{F}_X .

It is well-known that every \mathcal{ML} -definable frame class is closed under taking both disjoint unions and generated subframes (see [3, Theorem 3.14 (i), (ii)]). However this is not the case for every $\mathcal{ML}(\sqcup)$ -definable class, see Example A.1 in Appendix A.

Recall that a closed disjunctive \sqcup -clause is a formula of the form $\bigvee_{i \in I} \sqcup \varphi_i$, where, for each $i \in I$, $\varphi_i \in \mathcal{ML}$.

Definition 6. We denote by $\bigvee \sqcup \mathcal{ML}$ the set of all closed disjunctive \sqcup -clauses.

The following proposition follows directly by Propositions A.2 and A.3 in Appendix A.

Proposition 1. Every $\mathcal{ML}(\sqcup^+)$ -definable frame class is closed under taking generated subframes.

Now since, by Example A.1 $\mathcal{ML}(\sqcup^+)$ is not closed under taking disjoint unions and $\mathcal{ML}(\sqcup)$ is not closed under generated submodels, and since by Proposition A.2 $\mathcal{ML}(\sqcup^+) =_F \bigvee \sqcup \mathcal{ML}$, the following strict hierarchy follows.

Proposition 2. $\mathcal{ML} <_F \mathcal{ML}(\sqcup^+) =_F \bigvee \sqcup \mathcal{ML} <_F \mathcal{ML}(\sqcup)$. Moreover, the same holds when we restrict ourselves to elementary frame classes.

4 Goldblatt-Thomason-style Theorem for $\mathcal{ML}(\sqcup^+)$

In addition to disjoint unions and generated subframes, we introduce two more frame constructions. With the help of these four constructions, we first review the existing characterizations of \mathcal{ML} - and $\mathcal{ML}(\sqcup)$ -definability when restricted to the elementary frame classes. We then give a novel characterization of $\mathcal{ML}(\sqcup^+)$ -definability again restricted to the elementary frame classes.

Definition 7 (Bounded Morphism). *Given any two frames $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$, a function $f : W \rightarrow W'$ is a bounded morphism if it satisfies the following two conditions:*

(Forth) *If wRv , then $f(w)R'f(v)$.*

(Back) *If $f(w)R'v'$, then wRv and $f(v) = v'$ for some $v \in W$.*

If f is surjective, we say that \mathfrak{F}' is a bounded morphic image of \mathfrak{F} .

Definition 8 (Ultrafilter Extensions). *Let $\mathfrak{F} = (W, R)$ be a Kripke frame, and $\text{Uf}(W)$ denote the set of all ultrafilters on W . Define the binary relation R^{uc} on the set $\text{Uf}(W)$ as follows: $\mathcal{U}R^{\text{uc}}\mathcal{U}'$ iff $X \in \mathcal{U}'$ implies $m_R(X) \in \mathcal{U}$, for every $X \subseteq W$, where $m_R(X) := \{w \in W \mid wRw' \text{ for some } w' \in X\}$. The frame $\text{uc}\mathfrak{F} = (\text{Uf}(W), R^{\text{uc}})$ is called the ultrafilter extension of \mathfrak{F} .*

A frame class \mathbb{F} *reflects* ultrafilter extensions if $\text{uc}\mathfrak{F} \in \mathbb{F}$ implies $\mathfrak{F} \in \mathbb{F}$ for every frame \mathfrak{F} . It is well-known that every \mathcal{ML} - or $\mathcal{ML}(\sqcup)$ -definable frame class is closed under taking bounded morphic images and reflects ultrafilter extensions (cf. [3, Theorem 3.14, Corollary 3.16 and Exercise 7.1.2]).

Theorem 2 (Goldblatt-Thomason Theorems for \mathcal{ML} [9] & $\mathcal{ML}(\sqcup)$ [10]).

(i) *An elementary frame class is \mathcal{ML} -definable if and only if it is closed under taking bounded morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions.*

(ii) *An elementary frame class is $\mathcal{ML}(\sqcup)$ -definable if and only if it is closed under taking bounded morphic images and reflects ultrafilter extensions.*

In order to characterize $\mathcal{ML}(\sqcup^+)$ -definability of elementary frame classes, we need to introduce the following notion of *reflection of finitely generated subframes*: a frame class \mathbb{F} *reflects* finitely generated subframes whenever it is the case for all frames \mathfrak{F} that, if every finitely generated subframe of \mathfrak{F} is in \mathbb{F} , then $\mathfrak{F} \in \mathbb{F}$.⁴ The fact that every $\mathcal{ML}(\sqcup^+)$ -definable class reflects finitely generated subframes follows by Propositions 2 and A.4 (in Appendix A).

Proposition 3. *Every $\mathcal{ML}(\sqcup^+)$ -definable class of Kripke frames reflects finitely generated subframes.*

⁴ Closure under generated subframes and reflection of finitely generated subframes characterize the definability of hybrid logic with satisfaction operators and downarrow binder when restricted elementary frame classes [1, Theorem 26].

Whereas the original Goldblatt-Thomason Theorem for basic modal logic was proved via duality between algebras and frames [9], our proof of Goldblatt-Thomason-style Theorem modifies the model-theoretic proof given by van Benthem [2] for basic modal logic. The proof of the following theorem can be found in Appendix B (Theorem B.1).

Theorem 3. *Given any elementary frame class \mathbb{F} , the following are equivalent:*

- (i) \mathbb{F} is $\mathcal{ML}(\boxplus^+)$ -definable.
- (ii) \mathbb{F} is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.

5 Frame Definability in Logics with Team Semantics

We will first introduce the team-based logics of interest in this paper, i.e. modal logic with intuitionistic disjunction $\mathcal{ML}(\oplus)$, modal dependence logic \mathcal{MDL} , and extended modal dependence logic \mathcal{EMDL} . We will then show that with respect to frame definability all of these logics coincide. Finally, we will compare these logics to logics extended with the universal modality. We show that surprisingly, with respect to frame definability, $\mathcal{ML}(\boxplus^+)$ coincides with $\mathcal{ML}(\oplus)$. It then follows that $\mathcal{ML}(\boxplus^+)$ coincides also with \mathcal{MDL} and \mathcal{EMDL} .

5.1 Syntax and Semantics

A subset T of the domain of a Kripke model \mathfrak{M} is called a *team* of \mathfrak{M} . We will next define three variants of modal logic for which the semantics is defined not via pointed Kripke models (\mathfrak{M}, w) but Kripke models with teams (\mathfrak{M}, T) .

The syntax of modal logic with intuitionistic disjunction $\mathcal{ML}(\oplus)(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the grammar rule $\varphi ::= (\varphi \oplus \varphi)$. The syntax of modal dependence logic $\mathcal{MDL}(\Phi)$ and extended modal dependence logic $\mathcal{EMDL}(\Phi)$ is obtained by extending the syntax of $\mathcal{ML}(\Phi)$ by the following grammar rule for each $n \in \omega$:

$$\varphi ::= \text{dep}(\varphi_1, \dots, \varphi_n, \psi), \text{ where } \varphi_1, \dots, \varphi_n, \psi \in \mathcal{ML}(\Phi).$$

In the additional grammar rules above for \mathcal{MDL} , we require that $\varphi_1, \dots, \varphi_n, \psi \in \Phi$. The intuitive meaning of the (modal) dependence atom $\text{dep}(\varphi_1, \dots, \varphi_n, \psi)$ is that the truth value of the formula ψ is completely determined by the truth values of $\varphi_1, \dots, \varphi_n$. As before, if the underlying set Φ of atomic propositions is clear from the context, we drop “ (Φ) ”.

Before we define the team semantics for $\mathcal{ML}(\oplus)$, \mathcal{MDL} , and \mathcal{EMDL} , let us first introduce some notation that makes defining the semantics simpler.

Definition 9. *Let $\mathfrak{M} = (W, R, V)$ be a model and T and S teams of \mathfrak{M} . Define*

$$R[T] := \{w \in W \mid \exists v \in T(vRw)\} \text{ and } R^{-1}[T] := \{w \in W \mid \exists v \in T(wRv)\}.$$

For teams T and S of \mathfrak{M} , we write $T[R]S$ if $S \subseteq R[T]$ and $T \subseteq R^{-1}[S]$.

Thus, $T[R]S$ holds if and only if for every $w \in T$ there exists some $v \in S$ such that wRv , and for every $v \in S$ there exists some $w \in T$ such that wRv . We are now ready to define the team semantics for $\mathcal{ML}(\otimes)$, \mathcal{MDL} , and \mathcal{EMDL} . We use the symbol “ \models ” for team semantics instead of the symbol “ \Vdash ” which was used for Kripke semantics.

Definition 10. Let \mathfrak{M} be a Φ -model and T a team of \mathfrak{M} . The satisfaction relation $\mathfrak{M}, T \models \varphi$ for $\mathcal{ML}(\otimes)(\Phi)$, $\mathcal{MDL}(\Phi)$, and $\mathcal{EMDL}(\Phi)$ is defined as follows.

$$\begin{aligned} \mathfrak{M}, T \models p &\Leftrightarrow w \in V(p) \text{ for every } w \in T. \\ \mathfrak{M}, T \models \neg p &\Leftrightarrow w \notin V(p) \text{ for every } w \in T. \\ \mathfrak{M}, T \models (\varphi \wedge \psi) &\Leftrightarrow \mathfrak{M}, T \models \varphi \text{ and } \mathfrak{M}, T \models \psi. \\ \mathfrak{M}, T \models (\varphi \vee \psi) &\Leftrightarrow \mathfrak{M}, T_1 \models \varphi \text{ and } \mathfrak{M}, T_2 \models \psi \text{ for some } T_1 \text{ and } T_2 \\ &\text{such that } T_1 \cup T_2 = T. \\ \mathfrak{M}, T \models \Diamond \varphi &\Leftrightarrow \mathfrak{M}, T' \models \varphi \text{ for some } T' \text{ such that } T[R]T'. \\ \mathfrak{M}, T \models \Box \varphi &\Leftrightarrow \mathfrak{M}, T' \models \varphi, \text{ where } T' = R[T]. \end{aligned}$$

For $\mathcal{ML}(\otimes)$ we have the following additional clause:

$$\mathfrak{M}, T \models (\varphi \otimes \psi) \Leftrightarrow \mathfrak{M}, T \models \varphi \text{ or } \mathfrak{M}, T \models \psi.$$

For \mathcal{MDL} and \mathcal{EMDL} we have the following additional clause:

$$\begin{aligned} \mathfrak{M}, T \models \text{dep}(\varphi_1, \dots, \varphi_n, \psi) &\Leftrightarrow \forall w, v \in T : \bigwedge_{1 \leq i \leq n} (\mathfrak{M}, \{w\} \models \varphi_i \Leftrightarrow \mathfrak{M}, \{v\} \models \varphi_i) \\ &\text{implies } (\mathfrak{M}, \{w\} \models \psi \Leftrightarrow \mathfrak{M}, \{v\} \models \psi). \end{aligned}$$

We say that a formula φ of $\mathcal{ML}(\otimes)(\Phi)$ ($\mathcal{MDL}(\Phi)$ and $\mathcal{EMDL}(\Phi)$, respectively) is *valid in a Φ -model $\mathfrak{M} = (W, R, V)$* , and write $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, T \models \varphi$ holds for every team T of \mathfrak{M} . For formulae of \mathcal{ML} , the team semantics and the semantics defined via pointed models in the following sense coincide:

Proposition 4 ([18]). Let \mathfrak{M} be a Φ -model, T be a team of \mathfrak{M} , and w a point of \mathfrak{M} . Then, for every formula φ of $\mathcal{ML}(\Phi)$

$$\mathfrak{M}, T \models \varphi \Leftrightarrow \forall w \in T : \mathfrak{M}, w \Vdash \varphi,$$

and especially $\mathfrak{M}, \{w\} \models \varphi \Leftrightarrow \mathfrak{M}, w \Vdash \varphi$.

From Proposition 4 it follows that for every model \mathfrak{M} and formula φ of \mathcal{ML} , $\mathfrak{M} \Vdash \varphi$ iff $\mathfrak{M} \models \varphi$.

Proposition 5 (Downwards closure). Let φ be a formula of $\mathcal{ML}(\otimes)$ or \mathcal{EMDL} . Given a model \mathfrak{M} , and teams $S \subseteq T$ of \mathfrak{M} : $\mathfrak{M}, T \models \varphi$ implies $\mathfrak{M}, S \models \varphi$.

Definition 11. We say that the formulae $\varphi_1, \varphi_2 \in \mathcal{L} \in \{\mathcal{ML}(\otimes), \mathcal{EMDL}\}$ are *equivalent (in team semantics)*, and write $\varphi_1 \equiv_T \varphi_2$, if for every model \mathfrak{M} and every team T of \mathfrak{M} the equivalence $\mathfrak{M}, T \models \varphi_1 \Leftrightarrow \mathfrak{M}, T \models \varphi_2$ holds.

Recall the definition of \equiv_K from Section 2. When the subscript (K or T) of \equiv_K or \equiv_T is clear from the context (or when the two definitions coincide), we omit the subscript and write simply \equiv . Note that, by Proposition 4, for $\varphi, \psi \in \mathcal{ML}$ the equivalence $\varphi \equiv_K \psi \Leftrightarrow \varphi \equiv_T \psi$ holds.

5.2 Frame Definability in Team Semantics

Recall the definitions of frame definability from Section 3, and note that the definitions given there apply also to logics with team semantics. In [11] it was shown that the expressive powers of \mathcal{EMDL} and $\mathcal{ML}(\otimes)$ coincide. From this together with the fact that $\mathcal{MDL} =_F \mathcal{EMDL}$ (see Proposition C.1 in Appendix C), we obtain the following proposition.

Proposition 6. $\mathcal{ML} \leq_F \mathcal{MDL} =_F \mathcal{EMDL} =_F \mathcal{ML}(\otimes)$

Theorem 4. A frame class \mathbb{F} is $\mathcal{ML}(\otimes)$ -definable iff it is $\mathcal{ML}(\boxplus^+)$ -definable.

Proof. Let \mathbb{F} be a frame class. By Proposition 2, it suffices to show that \mathbb{F} is $\mathcal{ML}(\otimes)$ -definable iff it is $\bigvee \boxplus \mathcal{ML}$ -definable. “If” and “Only If” parts follow directly from Lemma C.2 and Lemma C.3 (in Appendix C), respectively. \square

We are finally ready to combine our results concerning frame definability of team-based modal logics and modal logics with the universal modality. By Propositions 2 and 6, and Theorem 4, we obtain the following strict hierarchy.

Theorem 5. $\mathcal{ML} <_F \mathcal{EMDL} =_F \mathcal{MDL} =_F \mathcal{ML}(\otimes) =_F \mathcal{ML}(\boxplus^+) <_F \mathcal{ML}(\boxplus)$. Moreover, the same holds when we restrict ourselves to the elementary frame classes.

We can now extend our Goldblatt-Thomason -style characterization (i.e., Theorem 3) to cover also the team-based logics \mathcal{MDL} , \mathcal{EMDL} , and $\mathcal{ML}(\otimes)$.

Corollary 1. For every logic $\mathcal{L} \in \{\mathcal{ML}(\boxplus^+), \mathcal{MDL}, \mathcal{EMDL}, \mathcal{ML}(\otimes)\}$ and for every elementary frame class \mathbb{F} , the following are equivalent:

- (i) \mathbb{F} is \mathcal{L} -definable.
- (ii) \mathbb{F} is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.

6 Conclusion

This paper initiated the study of frame definability in the context of team-based modal logics. We identified a connection between modal logics with team semantics and modal logic extended with the universal modality. We showed that, with respect to frame definability, we have the following strict hierarchy:

$$\mathcal{ML} <_F \mathcal{MDL} =_F \mathcal{EMDL} =_F \mathcal{ML}(\otimes) =_F \mathcal{ML}(\boxplus^+) <_F \mathcal{ML}(\boxplus).$$

In addition we gave a Goldblatt-Thomason -style characterization for the frame definability of \mathcal{MDL} , \mathcal{EMDL} , $\mathcal{ML}(\otimes)$, and $\mathcal{ML}(\boxplus^+)$. We showed that an elementary class of frames is definable in one (all) of those logics if and only if the class is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.

References

1. Areces, C., ten Cate, B.: Hybrid logics. In: Blackburn, P., van Benthem, J., Wolter, F. (eds.) *Handbook of Modal Logic*, pp. 821–868. Elsevier (2007).
2. van Benthem, J.: Modal frame classes revisited. *Fundamenta Informaticae* 18, 303–17 (1993).
3. Blackburn, P., de Rijke, M., Venema, Y.: *Modal Logic*. Cambridge University Press, New York, NY, USA (2001).
4. ten Cate, B.: Model theory for extended modal languages. Ph.D. thesis, University of Amsterdam, Institute for Logic, Language and Computation (2005).
5. Chang, C.C., Keisler, H.J.: *Model Theory*. North-Holland Publishing Company, Amsterdam, 3 edn. (1990).
6. Ebbing, J., Hella, L., Meier, A., Müller, J.S., Virtema, J., Vollmer, H.: Extended modal dependence logic. In: *WoLLIC*. pp. 126–137 (2013).
7. Ebbing, J., Lohmann, P., Yang, F.: Model checking for modal intuitionistic dependence logic. In: Bezhanishvili, G., Löbner, S., Marra, V., Richter, F. (eds.) *Logic, Language, and Computation, Lecture Notes in Computer Science*, vol. 7758, pp. 231–256. Springer (2013).
8. Gargov, G., Goranko, V.: Modal logic with names. *Journal of Philosophical Logic* 22, 607–36 (1993).
9. Goldblatt, R.I., Thomason, S.K.: Axiomatic classes in propositional modal logic. In: Crossley, J.N. (ed.) *Algebra and Logic*, pp. 163–73. Springer-Verlag (1975).
10. Goranko, V., Passy, S.: Using the universal modality: Gains and questions. *J. Log. Comput.* 2(1), 5–30 (1992).
11. Hella, L., Luosto, K., Sano, K., Virtema, J.: The expressive power of modal dependence logic. In: *AiML 2014* (2014).
12. Kontinen, J., Müller, J.S., Schnoor, H., Vollmer, H.: Modal independence logic. In: *AiML 2014* (2014).
13. Kontinen, J., Müller, J.S., Schnoor, H., Vollmer, H.: A van Benthem theorem for modal team semantics. *arXiv:1410.6648* (2014).
14. Lohmann, P., Vollmer, H.: Complexity results for modal dependence logic. *Studia Logica* 101(2), 343–366 (2013).
15. Müller, J.S., Vollmer, H.: Model checking for modal dependence logic: An approach through post’s lattice. In: *WoLLIC*, pp. 238–250 (2013).
16. Sano, K., Ma, M.: Goldblatt-Thomason-style theorems for graded modal language. In: Beklemishev, L., Goranko, V., Shehtman, V. (eds.) *Advances in Modal Logic* 2010. pp. 330–349, (2010).
17. Sano, K., Virtema, J.: Axiomatizing propositional dependence logics. *arXiv:1410.5038* (2014).
18. Sevenster, M.: Model-theoretic and computational properties of modal dependence logic. *J. Log. Comput.* 19(6), 1157–1173 (2009).
19. Virtema, J.: Complexity of validity for propositional dependence logics. In: *GandALF 2014* (2014).
20. Väänänen, J.: Modal dependence logic. In: Apt, K.R., van Rooij, R. (eds.) *New Perspectives on Games and Interaction, Texts in Logic and Games*, vol. 4, pp. 237–254 (2008).
21. Yang, F.: On Extensions and Variants of Dependence Logic. Ph.D. thesis, University of Helsinki (2014).

A Modal logic with universal modality

Proposition A.1. *Let $\varphi, \psi \in \mathcal{ML}(\sqcup)$ such that ψ is closed. Then, (i) $\Box(\varphi \vee \psi) \equiv_K (\Box\varphi \vee \psi)$; (ii) $\Diamond(\varphi \wedge \psi) \equiv_K (\Diamond\varphi \wedge \psi)$; (iii) $\sqcup(\varphi \vee \psi) \equiv_K (\sqcup\varphi \vee \psi)$.*

Proof. (i) and (iii) follow from [10, Proposition 3.6]. (ii) is completely analogous to the item (i). \square

Theorem A.1. *For each $\mathcal{ML}(\sqcup^+)$ -formula φ , there exists a $\mathcal{ML}(\sqcup^+)$ -formula ψ in \sqcup -form such that $\varphi \equiv_K \psi$.*

Proof. The proof is done by induction on φ . The cases for literals and connectives are trivial. As for the case $\varphi = \Box\psi$, we proceed as follows. By induction hypothesis there exists a conjunctive \sqcup -form $\bigwedge_{i \in I} \psi_i$, where each ψ_i is a disjunctive \sqcup -clause, such that $\bigwedge_{i \in I} \psi_i \equiv_K \psi$. By the semantics of \Box , we have that $(\Box\psi \equiv_K) \Box \bigwedge_{i \in I} \psi_i \equiv_K \bigwedge_{i \in I} \Box\psi_i$. Now since each ψ_i is a disjunctive \sqcup -clause, it follows from item (i) of Proposition A.1 that, for each $i \in I$, the formula $\Box\psi_i$ is equivalent to some disjunctive \sqcup -clause ψ'_i . Thus $\bigwedge_{i \in I} \psi'_i$ is a conjunctive \sqcup -form that is equivalent to $\Box\psi$.

The proof for the case of $\sqcup\varphi$ is otherwise the same as the proof for the case $\Box\varphi$, but instead of item (i) of Proposition A.1, item (iii) is used. The proof for the case $\Diamond\varphi$ is likewise analogous to that of $\Box\varphi$. The proof uses a disjunctive \sqcup -form instead of the conjunctive one and item (ii) of Proposition A.1 instead of item (i). \square

Example A.1. Consider the following examples from [10, p.14]: the formula $\neg p \vee \sqcup p$ defines the class $\{(W, R) \in \mathbb{F}_{\text{all}} \mid |W| = 1\}$, whereas the formula $\Diamond \Diamond(p \vee \neg p)$ defines the class $\{(W, R) \in \mathbb{F}_{\text{all}} \mid R \neq \emptyset\}$. Clearly, the former is not closed under taking disjoint unions, and the latter is not closed under taking generated subframes. Note that both of the classes above are elementary.

Lemma A.1. *For each $\mathcal{ML}(\sqcup^+)$ -formula φ , there exists a finite set Γ of closed disjunctive \sqcup -clauses such that $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \Gamma$ for every model \mathfrak{M} .*

Proof. Let φ be an $\mathcal{ML}(\sqcup^+)$ -formula. By Theorem 1, we may assume that φ is a conjunctive \sqcup -form $\bigwedge_{i \in I} \psi_i$, where each $\psi_i := \gamma_i \vee \bigvee_{j \in J_i} \sqcup \delta_j$ is a disjunctive \sqcup -clause. By Proposition A.1 (iii), for each $i \in I$, $\sqcup \psi_i$ is equivalent to the closed disjunctive \sqcup -clause $\psi'_i := \sqcup \gamma_i \vee \bigvee_{j \in J_i} \sqcup \delta_j$. Thus, for every model \mathfrak{M} ,

$$\mathfrak{M} \models \bigwedge_{i \in I} \psi_i \Leftrightarrow \mathfrak{M} \models \{\psi_i \mid i \in I\} \Leftrightarrow \mathfrak{M} \models \{\sqcup \psi_i \mid i \in I\} \Leftrightarrow \mathfrak{M} \models \{\psi'_i \mid i \in I\}.$$

Proposition A.2. $\mathcal{ML}(\sqcup^+) =_F \bigvee \sqcup \mathcal{ML}$.

Proof. The direction $\bigvee \sqcup \mathcal{ML} \leq_F \mathcal{ML}(\sqcup^+)$ is trivial. We show that $\mathcal{ML}(\sqcup^+) \leq_F \bigvee \sqcup \mathcal{ML}$. Consider any $\mathcal{ML}(\sqcup^+)$ -definable class of frames \mathbb{F} . Let Γ be a set of $\mathcal{ML}(\sqcup^+)$ formulae that defines \mathbb{F} . By Lemma A.1, for each $\varphi \in \Gamma$, there is a finite set Δ_φ of closed disjunctive \sqcup -clauses such that $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \Delta_\varphi$ for every Kripke model \mathfrak{M} . It follows that $\mathfrak{M} \models \Gamma$ iff $\mathfrak{M} \models \bigcup_{\varphi \in \Gamma} \Delta_\varphi$ for every Kripke model \mathfrak{M} . Therefore, $\bigcup_{\varphi \in \Gamma} \Delta_\varphi$ also defines \mathbb{F} , as desired. \square

Proposition A.3. *Let \mathfrak{F} be a frame and φ a closed disjunctive \sqcup -clause. If $\mathfrak{F} \Vdash \varphi$, then $\mathfrak{G} \Vdash \varphi$ for all generated subframes \mathfrak{G} of \mathfrak{F} .*

Proof. Fix any generated subframe \mathfrak{G} of a frame \mathfrak{F} and put $\varphi := \bigvee_{i \in I} \sqcup \psi_i$. Suppose that $\mathfrak{F} \Vdash \varphi$. To show $\mathfrak{G} \Vdash \varphi$, fix any valuation V and any state w in \mathfrak{G} . We show that $(\mathfrak{G}, V), w \Vdash \sqcup \psi_i$ for some $i \in I$. Since we can regard V as a valuation on \mathfrak{F} , $(\mathfrak{F}, V), w \Vdash \bigvee_{i \in I} \sqcup \psi_i$. Thus there is some $i \in I$ such that $(\mathfrak{F}, V), u \Vdash \psi_i$, for every $u \in |\mathfrak{F}|$. Fix such $i \in I$. Since ψ_i is in \mathcal{ML} and the satisfaction of \mathcal{ML} is invariant under taking generated submodels (cf. [3, Proposition 2.6]), $(\mathfrak{G}, V), u \Vdash \psi_i$ for every $u \in \mathfrak{G}$. Therefore, $(\mathfrak{G}, V), w \Vdash \sqcup \psi_i$, as desired. \square

Proposition A.4. *Let \mathfrak{F} be a frame and φ a closed disjunctive \sqcup -clause. If $\mathfrak{G} \Vdash \varphi$ for all finitely generated subframes \mathfrak{G} of \mathfrak{F} , then $\mathfrak{F} \Vdash \varphi$.*

Proof. We show the contrapositive implication. Let φ be $\bigvee_{i \in I} \sqcup \psi_i$ and suppose that $\mathfrak{F} \not\Vdash \bigvee_{i \in I} \sqcup \psi_i$. Now, we can find a valuation V and a state w such that $(\mathfrak{F}, V), w \not\Vdash \sqcup \psi_i$ for all $i \in I$. Thus, for each $i \in I$, there is a state w_i such that $(\mathfrak{F}, V), w_i \not\Vdash \psi_i$. Define $X := \{w_i \mid i \in I\}$ and note that X is finite. Consider the submodel (\mathfrak{F}_X, V_X) of \mathfrak{F} generated by X . Since for each $i \in I$, $(\mathfrak{F}, V), w_i \not\Vdash \psi_i$ and $\psi_i \in \mathcal{ML}$, and since the satisfaction of \mathcal{ML} is invariant under generated submodels (cf. [3, Proposition 2.6]), it follows that $(\mathfrak{F}_X, V_X), w_i \not\Vdash \psi_i$ for each $i \in I$. Thus $(\mathfrak{F}_X, V_X) \not\Vdash \sqcup \psi_i$ for each $i \in I$. Hence $(\mathfrak{F}_X, V_X) \not\Vdash \bigvee_{i \in I} \sqcup \psi_i$, which implies our goal $\mathfrak{F}_X \not\Vdash \bigvee_{i \in I} \sqcup \psi_i$. \square

B Goldblatt-Thomason Theorem

Definition B.1 (Satisfiability). *Let Γ be a set of formulae, \mathfrak{M} a model and \mathbb{F} a class of frames. We say that Γ is satisfiable in \mathfrak{M} if there exists a point w of \mathfrak{M} such that $\mathfrak{M}, w \Vdash \gamma$ for all $\gamma \in \Gamma$. We say that Γ is finitely satisfiable in \mathfrak{M} if each finite subset of Γ is satisfiable in \mathfrak{M} . We say that Γ is satisfiable in \mathbb{F} if there exists a frame $\mathfrak{F} \in \mathbb{F}$ and a valuation V on \mathfrak{F} such that Γ is satisfiable in (\mathfrak{F}, V) . Finally, we say that Γ is finitely satisfiable in \mathbb{F} if each finite subset of Γ is satisfiable in \mathbb{F} .*

Theorem B.1. *Given any elementary frame class \mathbb{F} , the following are equivalent:*

- (i) \mathbb{F} is $\mathcal{ML}(\sqcup^+)$ -definable.
- (ii) \mathbb{F} is closed under taking generated subframes and bounded morphic images, and reflects ultrafilter extensions and finitely generated subframes.

Proof. The direction from (i) to (ii) follows directly by Propositions 1 and 3, and Theorem 2. In the proof of the converse direction, we use some notions from first-order model theory such as elementary extensions and ω -saturation. The reader unfamiliar with them is referred to [5]. Assume (ii) and define $\text{Log}(\mathbb{F}) := \{\varphi \in \mathcal{ML}(\sqcup^+) \mid \mathbb{F} \Vdash \varphi\}$. We show that, for any frame \mathfrak{F} , $\mathfrak{F} \in \mathbb{F}$ iff $\mathfrak{F} \Vdash \text{Log}(\mathbb{F})$.

Consider any $\mathfrak{F} = (W, R)$. It is trivial to show the Only-If-direction, and so we show the If-direction. Assume that $\mathfrak{F} \models \text{Log}(\mathbb{F})$. To show $\mathfrak{F} \in \mathbb{F}$, we may assume, without loss of generality, that \mathfrak{F} is finitely generated. This is because: otherwise, it would suffice to show, since \mathbb{F} reflects finitely generated subframes, that $\mathfrak{G} \in \mathbb{F}$ for all finitely generated subframes \mathfrak{G} of \mathfrak{F} . Let U be a finite generator of \mathfrak{F} . Let us expand our syntax with a (possibly uncountable) set $\{p_A \mid A \subseteq W\}$ of new propositional variables and define Δ to be the set containing exactly:

$$p_{A \cap B} \leftrightarrow p_A \wedge p_B, \quad p_{W \setminus A} \leftrightarrow \neg p_A, \quad p_{m_R(A)} \leftrightarrow \Diamond p_A, \quad p_W,$$

where $A, B \subseteq W$ and $m_R(A) := \{x \in W \mid xRy \text{ for some } y \in A\}$. Define

$$\Delta_{\mathfrak{F}, u} := \{p_{\{u\}} \wedge \bigwedge_{\substack{i \leq n \\ \varphi \in \Delta}} \Box^i \varphi \mid n \in \omega, \Delta \text{ is finite}\}, \quad \Delta_{\mathfrak{F}} := \{\Diamond \varphi \mid \varphi \in \bigcup_{u \in U} \Delta_{\mathfrak{F}, u}\},$$

for each $u \in U$. Note that all formulae in $\Delta_{\mathfrak{F}}$ are $\mathcal{ML}(\Box)$ -formulae and that each formula in $\Delta_{\mathfrak{F}, u}$ belong to \mathcal{ML} . Recall that \mathfrak{F} is finitely generated by U . The intuition here is that $\Delta_{\mathfrak{F}}$ provides a “complete enough description” of \mathfrak{F} .

We will now show that $\Delta_{\mathfrak{F}}$ is satisfiable in \mathbb{F} . Since \mathbb{F} is elementary, it follows from the compactness of first-order logic that it suffices to show that $\Delta_{\mathfrak{F}}$ is finitely satisfiable in \mathbb{F} . Let Γ be a finite subset of $\Delta_{\mathfrak{F}}$ and write $\Gamma := \{\Diamond \gamma_1, \dots, \Diamond \gamma_n\}$. We note that each γ_i belongs to \mathcal{ML} . Assume, for the sake of a contradiction, that $\mathbb{F} \not\models \neg \bigwedge_{1 \leq i \leq n} \Diamond \gamma_i$. Now clearly $\mathbb{F} \models \vartheta$, where $\vartheta := \bigvee_{1 \leq i \leq n} \Box \neg \gamma_i$. Since ϑ is an $\mathcal{ML}(\Box^+)$ -formula, it belongs to $\text{Log}(\mathbb{F})$. Thus by the assumption $\mathfrak{F} \models \text{Log}(\mathbb{F})$, we conclude that $\mathfrak{F} \models \vartheta$, and therefore $\mathfrak{F} \models \neg \bigwedge_{1 \leq i \leq n} \Diamond \gamma_i$. However, $\bigwedge \Gamma$ is clearly satisfiable in \mathfrak{F} under the natural valuation sending p_A to A , which implies $\mathfrak{F} \not\models \neg \bigwedge \Gamma$. This is a contradiction. Therefore, $\Delta_{\mathfrak{F}}$ is satisfiable in \mathbb{F} .

Let $\mathfrak{H} \in \mathbb{F}$ be such that $\Delta_{\mathfrak{F}}$ is satisfiable in \mathfrak{H} . Let us fix a valuation V such that $(\mathfrak{H}, V) \models \Delta_{\mathfrak{F}}$. Let (\mathfrak{H}^*, V^*) be some ω -saturated elementary extension of (\mathfrak{H}, V) . Clearly also $(\mathfrak{H}^*, V^*) \models \Delta_{\mathfrak{F}}$. It then follows that, for each $u \in U$, $\Delta_{\mathfrak{F}, u}$ is finitely satisfiable in (\mathfrak{H}^*, V^*) . Thus, by ω -saturatedness, $\Delta_{\mathfrak{F}, u}$ is satisfiable in (\mathfrak{H}^*, V^*) for each $u \in U$. For each $u \in U$, let w_u denote a point such that $(\mathfrak{H}^*, V^*), w_u \models \Delta_{\mathfrak{F}, u}$. Define $Z := \{w_u \mid u \in U\}$. Now let $(\mathfrak{G}_Z^*, V_Z^*)$ denote some ω -saturated elementary extension of the Z generated submodel of (\mathfrak{H}^*, V^*) . It is easy to check that $(\mathfrak{G}_Z^*, V_Z^*) \models \Delta_{\mathfrak{F}}$ and $(\mathfrak{G}_Z^*, V_Z^*) \models \Delta$. Since \mathbb{F} is elementary and closed under taking generated subframes, we conclude first that $\mathfrak{H}^* \in \mathbb{F}$ and then that $\mathfrak{G}_Z^* \in \mathbb{F}$. We can now prove the following claim.

Claim. The ultrafilter extension $\text{uc}\mathfrak{F}$ is a bounded morphic image of \mathfrak{G}_Z^* .

By closure of \mathbb{F} under bounded morphic images, we obtain $\text{uc}\mathfrak{F} \in \mathbb{F}$. Finally, since \mathbb{F} reflects ultrafilter extensions, $\mathfrak{F} \in \mathbb{F}$, as required. \square

(Proof of Claim) Define a mapping $f : |\mathfrak{G}_Z^*| \rightarrow \text{Uf}(W)$ (where $\text{Uf}(W)$ is the set of all ultrafilters on W) by

$$f(s) := \{A \subseteq W \mid (\mathfrak{G}_Z^*, V_Z^*), s \models p_A\}.$$

We will show that (a) $f(s)$ is an ultrafilter on W ; (b) f is a bounded morphism; (c) f is surjective. Below, we denote by S the underlying binary relation of \mathfrak{G}_Z^* .

- (a) $f(u)$ is an ultrafilter: Follows immediately from the fact that $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Delta$.
- (b1) f satisfies **(Forth)**: We show that sSs' implies $f(s)R^{uc}f(s')$. Assume that sSs' . By the definition of R^{uc} , it suffices to show that $A \in f(s')$ implies $m_R(A) \in f(s)$. Suppose $A \in f(s')$. Thus $(\mathfrak{G}_Z^*, V_Z^*), s' \Vdash p_A$. Since sSs' , we obtain $(\mathfrak{G}_Z^*, V_Z^*), s \Vdash \Diamond p_A$. Since $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Delta$, $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Diamond p_A \leftrightarrow p_{m_R(A)}$. Therefore $(\mathfrak{G}_Z^*, V_Z^*), s \Vdash p_{m_R(A)}$, and hence $m_R(A) \in f(s)$, as desired.
- (b2) f satisfies **(Back)**: We show that $f(s)R^{uc}\mathcal{U}$ implies sSs' and $f(s') = \mathcal{U}$ for some $s' \in |\mathfrak{G}_Z^*|$. Assume that $f(s)R^{uc}\mathcal{U}$. We will find a state s' such that sSs' and $(\mathfrak{G}_Z^*, V_Z^*), s' \Vdash p_A$ for all $A \in \mathcal{U}$. By ω -saturation, it suffices to show that $\{p_A \mid A \in \mathcal{U}\}$ is finitely satisfiable in the set $\{t \in |\mathfrak{G}_Z^*| \mid sSt\}$ of the successors of s . Take any $A_1, \dots, A_n \in \mathcal{U}$. Then, $\bigcap_{1 \leq i \leq n} A_i \in \mathcal{U}$. Now since $f(s)R^{uc}\mathcal{U}$, $m_R(\bigcap_{1 \leq i \leq n} A_i) \in f(s)$. Hence $(\mathfrak{G}_Z^*, V_Z^*), s \Vdash p_{m_R(\bigcap_{1 \leq i \leq n} A_i)}$. Since $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Delta$, $(\mathfrak{G}_Z^*, V_Z^*) \Vdash p_{m_R(\bigcap_{1 \leq i \leq n} A_i)} \leftrightarrow \Diamond p_{\bigcap_{1 \leq i \leq n} A_i}$. Therefore $(\mathfrak{G}_Z^*, V_Z^*), s \Vdash \Diamond p_{\bigcap_{1 \leq i \leq n} A_i}$. Thus there is a state $s' \in |\mathfrak{G}_Z^*|$ such that sSs' and $(\mathfrak{G}_Z^*, V_Z^*), s' \Vdash p_{\bigcap_{1 \leq i \leq n} A_i}$. Therefore and since $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Delta$, it follows that $(\mathfrak{G}_Z^*, V_Z^*), s' \Vdash p_{A_i}$ for all $1 \leq i \leq n$.
- (c) f is surjective: Let us take any ultrafilter $\mathcal{U} \in |\mathfrak{uc}\mathfrak{F}|$. To prove surjectiveness, we show that the set $\{p_A \mid A \in \mathcal{U}\}$ is satisfiable in $(\mathfrak{G}_Z^*, V_Z^*)$. By ω -saturatedness of $(\mathfrak{G}_Z^*, V_Z^*)$, it suffices to show finite satisfiability. Fix any $A_1, \dots, A_n \in \mathcal{U}$. It follows that $\bigcap_{1 \leq k \leq n} A_k \in \mathcal{U}$, and hence $\bigcap_{1 \leq k \leq n} A_k \neq \emptyset$. Pick $w \in \bigcap_{1 \leq k \leq n} A_k$. Since \mathfrak{F} is finitely generated by U , w is reachable (in \mathfrak{F}) from some point $u \in U$ in a finite number of steps. But then there is some $l \in \omega$ such that $(\mathfrak{F}, V_0), u \Vdash p_{(m_R)^l(\bigcap_{1 \leq k \leq n} A_k)}$, where V_0 is the natural valuation on \mathfrak{F} sending p_X to X . This implies that $(\mathfrak{F}, V_0) \Vdash \Diamond(p_{\{u\}} \wedge p_{(m_R)^l(\bigcap_{1 \leq k \leq n} A_k)})$. Since V_0 is the natural valuation, we also obtain that $u \in (m_R)^l(\bigcap_{1 \leq k \leq n} A_k)$, and thus Δ contains $p_{\{u\}} \leftrightarrow p_{\{u\}} \wedge p_{(m_R)^l(\bigcap_{1 \leq k \leq n} A_k)}$. Therefore and since $\Delta_{\mathfrak{F}} \Vdash \Diamond p_{\{u\}}$, $\Delta_{\mathfrak{F}} \Vdash \Diamond(p_{\{u\}} \wedge p_{(m_R)^l(\bigcap_{1 \leq k \leq n} A_k)})$. It now follows from $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Delta_{\mathfrak{F}}$ that $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Diamond(p_{\{u\}} \wedge p_{(m_R)^l(\bigcap_{1 \leq k \leq n} A_k)})$. Hence there is a state s in \mathfrak{G}_Z^* such that $(\mathfrak{G}_Z^*, V_Z^*), s \Vdash p_{(m_R)^l(\bigcap_{1 \leq k \leq n} A_k)}$. Since $(\mathfrak{G}_Z^*, V_Z^*) \Vdash \Delta$, $(\mathfrak{G}_Z^*, V_Z^*), s \Vdash \Diamond^l p_{\bigcap_{1 \leq k \leq n} A_k}$. Therefore, $\{p_{A_1}, \dots, p_{A_n}\}$ is satisfiable in $(\mathfrak{G}_Z^*, V_Z^*)$. \dashv

C Modal logics with team semantics

Proposition C.1. *Let Φ be an infinite set of proposition symbols. For every formula $\varphi \in \mathcal{EMDL}(\Phi)$ there exists a formula $\varphi^* \in \mathcal{MDL}(\Phi)$ such that $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models \varphi^*$ for every frame \mathfrak{F} .*

Proof. We give a sketch of the proof here. A detailed proof is given in the extended version of [19] (to appear). The translation $\varphi \mapsto \varphi^*$ is defined inductively in the following way. For (negated) proposition symbols the translation is the identity. For propositional connectives and modalities we define

$$(\psi_1 \oplus \psi_2) \mapsto (\psi_1^* \oplus \psi_2^*), \quad \text{and} \quad \nabla \psi \mapsto \nabla \psi^*,$$

where $\oplus \in \{\wedge, \vee\}$ and $\nabla \in \{\Diamond, \Box\}$. The only nontrivial case is the case for the dependence atoms. Let φ be the dependence atom $\text{dep}(\psi_1, \dots, \psi_n)$, let k be the modal depth of φ , and let p_1, \dots, p_n be distinct fresh proposition symbols. Define

$$\varphi^* := \left(\bigwedge_{0 \leq i \leq k} \Box^i \bigwedge_{1 \leq j \leq n} (p_j \leftrightarrow \psi_j) \right) \rightarrow \text{dep}(p_0, \dots, p_n).$$

It is now straightforward to show that the claim follows.

Definition C.1. We say that an $\mathcal{ML}(\otimes)$ -formula φ is in \otimes -normal form if $\varphi = \psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_n$ for some $n \in \omega$ and $\psi_1, \psi_2, \dots, \psi_n \in \mathcal{ML}(\Phi)$.

Proposition C.2 (\otimes -normal form, [19, 21]). For every $\mathcal{ML}(\otimes)$ -formula φ there exists an equivalent formula in \otimes -normal form.

Lemma C.1. For every \mathcal{ML} -formula φ and model \mathfrak{M} : $\mathfrak{M} \models \Box \varphi$ iff $\mathfrak{M}, W \models \varphi$.

Proof. By the semantics of \Box , $\mathfrak{M} \models \Box \varphi$ iff $\mathfrak{M}, w \models \varphi$ for every $w \in W$. Furthermore by Proposition 4, $\mathfrak{M}, w \models \varphi$ for every $w \in W$ iff $\mathfrak{M}, W \models \varphi$. \square

Lemma C.2. For every $\mathcal{ML}(\otimes)$ -formula φ there exists a closed disjunctive \Box -clause φ^- such that $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi^-$ for every Kripke model \mathfrak{M} .

Proof. Let φ be an arbitrary $\mathcal{ML}(\otimes)$ -formula. By Proposition C.2, we may assume that $\varphi = \psi_1 \otimes \dots \otimes \psi_n$, for some $n \in \omega$ and $\psi_1, \dots, \psi_n \in \mathcal{ML}$. Let $\mathfrak{M} = (W, R, V)$ be an arbitrary model. It suffices to show $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M} \models \Box \psi_1 \vee \dots \vee \Box \psi_n$. This is shown as follows.

$$\begin{array}{lll} \mathfrak{M} \models \varphi & \begin{array}{c} \text{Def. of } \models \\ \text{Proposition 5} \\ \Leftrightarrow \\ \text{Def. of } \otimes \\ \Leftrightarrow \\ \text{Lemma C.1} \\ \Leftrightarrow \\ \text{Defs. of } \models, \Box \text{ and } \vee \end{array} & \begin{array}{l} \mathfrak{M}, W \models \psi_1 \otimes \dots \otimes \psi_n \\ \text{There exists } i \leq n: \mathfrak{M}, W \models \psi_i \\ \text{There exists } i \leq n: \mathfrak{M} \models \Box \psi_i \\ \mathfrak{M} \models \Box \psi_1 \vee \dots \vee \Box \psi_n. \end{array} \end{array}$$

Lemma C.3. For every closed disjunctive \Box -clause $\varphi \in \mathcal{ML}(\Box^+)$ there exists an $\mathcal{ML}(\otimes)$ -formula φ^* such that $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi^*$ for every Kripke model \mathfrak{M} .

Proof. Let φ be an arbitrary closed disjunctive \Box -clause, i.e., $\varphi = \Box \psi_1 \vee \dots \vee \Box \psi_n$ for some $n \in \omega$ and $\psi_1, \dots, \psi_n \in \mathcal{ML}$. Let $\mathfrak{M} = (W, R, V)$ be an arbitrary Kripke model. It suffices to show $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M} \models \psi_1 \otimes \dots \otimes \psi_n$. We proceed as follows.

$$\begin{array}{lll} \mathfrak{M} \models \Box \psi_1 \vee \dots \vee \Box \psi_n & \begin{array}{c} \text{Defs. of } \models, \Box, \text{ and } \vee \\ \Leftrightarrow \\ \text{Lemma C.1} \\ \Leftrightarrow \\ \text{Def. of } \otimes \\ \Leftrightarrow \\ \text{Proposition 5} \end{array} & \begin{array}{l} \text{There exists } i \leq n: \mathfrak{M} \models \Box \psi_i \\ \text{There exists } i \leq n: \mathfrak{M}, W \models \psi_i. \\ \mathfrak{M}, W \models \psi_1 \otimes \dots \otimes \psi_n \\ \mathfrak{M} \models \psi_1 \otimes \dots \otimes \psi_n. \end{array} \end{array}$$